

Entanglement and Thermodynamics of Black Hole Entropy

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Abstract

Using simple conditions drawn from the stability of the cosmos in terms of vacuum energy density, the cut-off momentum of entanglement is related to the planckian mass. In so doing the black hole entropy is shown to be independent of the number of field species that contribute to vacuum fluctuations. And this is in spite of the fact that the number of field species is a linear multiplicand of the entanglement entropy when this latter is expressed in terms of the fundamental momentum cut-off of all fields.

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An interesting problem of fundamental character arises upon comparing black hole entropy, S , as determined thermodynamically by Hawking [1] from black hole evaporation, and a statistical procedure called entanglement entropy [2, 3]. The two ought to be the same if they do indeed describe the same physics. The formulae to be compared are:

$$S = m_{pl}^2 A \quad (\text{Hawking}) \quad (1a)$$

$$= \nu \Lambda^2 A \quad (\text{Entanglement}) \quad (1b)$$

All constants of $O(1)$ are set equal to 1. A is the area of the black hole horizon, m_{pl} = planck mass, Λ is a momentum cut-off which has been introduced to implement the counting procedure that leads to equation (1b). Also, ν is the number of species of fields which is introduced to describe the field vacuum fluctuations.

One's first inclination would have been to set $\Lambda = m_{pl}$ since m_{pl}^{-1} is the fundamental length scale of gravitation; and indeed of all physics. Since it is thought to describe geometry, $m_{pl}^2 R \sqrt{g}$ being the gravitational action without reference to matter, m_{pl} should contain no reference to ν . Therefore, consistency requires that we give up our first inclination and attribute a dynamical character to Λ wherein

$$\Lambda^2 = m_{pl}^2 / \nu. \quad (2)$$

This short essay contains a simple argument which we believe has the germs of a rigorous derivation of equation (2). It is to be noted that equation (2) has far reaching implications in our appreciation of quantum field theory. Λ is not a momentum cut-off which has been conveniently introduced as a regulator to make calculations possible. Rather it is a dynamical parameter precisely determined through use of the laws of physics. We shall show that it comes about from the existence of a stable cosmos.

Let us begin by reconsidering the origins of equations (1a),(1b). First, equation (1a) follows from the existence of a black hole temperature, a concept that arises from the periodicity of asymptotic field Green's functions in imaginary time. The temperature is the inverse of that period. Dimensional arguments suffice to deduce $T_{BH} = m_{pl}^2 / M$ (T_{BH} = black hole temperature; M = the black hole mass). The radius of the black hole is M / m_{pl}^2 , hence $A = M^2 / m_{pl}^4$. Integration of $dS = dE / T = dM / T$ then gives equation (1a).

Equation (1b) is conveniently derived from the partially traced density matrix that describes field configurations about the black hole. Since S is asymptotic in character (i.e., far from the black hole) it suffices to consider field modes in flat space. Therefore one can idealize. Consider a cubic lattice of cells Λ^{-3} in dimension. Divide the cube into two parts, large and small, by a plane at, say $Z = \text{constant}$. The field can be modelled as a set of springs coupling neighbouring points, so we have left (L) and right (R) field configurations.

To seize the meaning of entanglement entropy first imagine removing all the springs directed along the Z axis that are bisected by the dividing plane. Then, L and R are decoupled and the Schrödinger representation of the ground state factorizes into two ground states, $\Psi = \Psi_L \Psi_R$ where, $\Psi_L(\Psi_R)$ refer to degrees of freedom in the $L(R)$ sectors. Then $S = S_L + S_R = 0 + 0$.

A non trivial entropy can be constructed by reinstating the coupling of L and R by replacing the missing springs and forming a reduced density matrix, ρ_R , by tracing over the L degrees of freedom, i.e., $\rho_R = \text{tr}_L \rho$, where tr_L is the trace over L degrees of freedom. The partial trace is carried out to express one's ignorance of the field configurations within the

black hole. Because of the existence of the springs at the LR boundary, the entropy S_R no longer vanishes, where $S_R = -tr_R \rho_R \ln \rho_R$. It is called entanglement entropy, the L degrees of freedom being inevitably tangled with those on the right because of the “bridge” springs across the boundary.

The number of such bridge springs per particle species is $\Lambda^2 A$ so the induced effect is expected to be $S_R = \nu \Lambda^2 A$. The factor ν arises from the ν species appearing in the trace. The argument of proportionality to A is correct, as such, only if there are no long range correlations. This is true if the particle at each site has a mass. A rigorous calculation [2,3,4] however shows that a mass is, in fact, unnecessary to complete the calculation and equation (1b) is, in general, the correct answer. As the details of this proof are not germane to our present purpose we refer the reader to the references.

The point of this note is to show how simple, yet fundamental, reasoning leads to the consistency condition (2). It is based essentially on the stability of the cosmos as deduced from quantum field theory (QFT) and general relativity (GR) applied to homogeneous flat spaces. These are the spaces that one generally considers asymptotically far from the black hole.

We begin with a highly oversimplified estimate for the vacuum energy density of such spaces for bosonic fields, based on the classification of field configurations in terms of modes. There are ν elementary fields, each developed in terms of modes and all cut-off at a common value of Λ (once more all factors $O(1)$ are set = 1).

$$\rho = \nu \Lambda^4 - \frac{\nu^2 \Lambda^6}{m_{pl}^2} \quad (3)$$

The first term on the right hand side is the zero point energy calculated to the lowest non-vanishing order in m_{pl}^{-2} (i.e. independent of m_{pl}). In addition there is the universal interaction among all fields, mediated by gravity. Since this effective interaction is attractive, the corresponding energy is negative. A crude approximation is a pairwise interaction, such as the Newtonian potential, as expressed by the second term on the RHS of Eq.3, to order $O(m_{pl}^{-2})$. Note that all species interact universally through gravity, hence the combinatorial factor of ν^2 , with the minus sign expressing the attractiveness of gravity. The dependence on Λ follows from dimensional arguments.

In the adiabatic era one has $\rho_{\text{Total}} = \rho_0 + \rho_M$ where ρ_M is the energy-density due to on-mass-shell quanta and ρ_0 is the vacuum energy, associated by most physicists, with dark energy. Whether ρ_0 is strictly positive or a quantity that fluctuates about zero mean, it cannot, in absolute value, exceed H^2 in order of magnitude. At the present time this is $O(10^{-100} m_{pl}^4)$ whereas each of the two contributions to the r.h.s. of Eq.3 are $O(m_{pl}^4/\nu)$. Therefore, in the adiabatic era, in good approximation the separate terms contributing to ρ_0 , given by Eq.3, cancel. And Eq.2 is secured. If a black hole during inflation the situation could be more complicated and will not be discussed here.

Before trying to evaluate the validity of the estimate given by Eq.3, it is meet that the reader appreciate the deep cosmological significance inherent in equation (3). To this end, it is convenient to envision $N(\Lambda) = \text{const.} \Lambda^3$, the density of modes, as an analog to the particle density of a quantum fluid which is self interacting through an attractive pairwise interaction. Whereas in conventional quantum fluids the ground state and its various concomitant dynamical parameters are determined variationally, this is not true of the “cosmic

fluid". Rather, the usual appeal to Pauli repulsion, which prevents total collapse, is replaced by the positivity condition, $\rho \geq 0$. As we have pointed out, in the adiabatic era this is tantamount to $\rho = 0$, thereby leading to equation (2). In a longer follow-up paper, these considerations will be extended to lead to further understanding of mode dilution, fluctuations such as dark energy, inflation and its fluctuations, and other features of cosmology. For the nonce, we merely wish to convey the message of equation (2), a highly nontrivial condition based on cosmic stability.

Let us now delve somewhat into the nature of the approximations inherent in Eqs.1a,1b,3. Equation 1a) is in the nature of a thermodynamic identity given the classical black hole metric, i.e., the neglect of backreaction occasioned by evaporation as well as the possible effect of fluctuations of the horizon. That the temperature is unaffected by self-interaction of the field is a well-known theorem. However, this has not been checked in the case that the interactions are gravitational. This essay is not concerned with these questions and Eq.1a is accepted as such.

Equation 1b) has been calculated using free field theory [2,3]. Hence its statute is different from equation 1a). To compare equations 1a) and 1b) is, therefore, analogous to the fleshing out of a thermodynamic identity with a formula derived from a kinetic model, here free field theory. Equation 2) is thus to be regarded as a consistency condition subject to the qualification that the free field theory is applicable to entanglement.

Accordingly, one should not push Eq.3 too hard. Each of the two terms is estimated to lowest order in m_{pl}^{-2} and one expects there to be corrections. It is also to be mentioned that the identification of Λ with a sharp momentum cutoff is an over-idealization. Rather, one expects Λ to be "fuzzy". This is because the considerations leading to Eq.3 show that Λ is determined from an equilibrium between zero point energy of modes and their gravitational interaction energy. This equilibrium fluctuates, hence the "fuzziness". On this basis, exact agreement of Eq.1a and Eq.1b to terms of $O(1)$ must be imposed on grounds of thermodynamic consistency. Therefore, this essay is to be considered only as a semi-quantitative explanation of Equation 2.

If $\nu \gg 1$, an expansion in ν^{-1} can be carried out. This must be done for Eq.1b as well as each of the terms contributing to Eq.2. Of these, the interaction term of Eq.2 is immediately evaluated to leading order in ν^{-1} , being a sum on simple loops. Each loop carries a factor $\nu m_{pl}^{-2} \Lambda^2$. This, being $O(1)$ does not affect the previous estimate at the precision given. It may be conjectured that the same is true for the other terms, but this remains to be carried out.

We close this essay with some relevant references. Parentani [5], in a series of works has shown that modes of sufficiently short wavelength are over-damped due to their scattering from vacuum fluctuations, thereby corroborating the idea that Λ is dynamically generated.

Effects of renormalization are extensively discussed by Jacobson and collaborators. See [6] and further references therein.

Susskind and Uglum [7] have investigated black hole entropy through use of the deficit angle formalism.

It will be interesting to interrelate these various approaches to the simple physical picture put forth in this essay.

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